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ON NONRELATIVISTIC $Q\bar{Q}$ POTENTIAL VIA THE WILSON LOOP IN GALILEAN SPACETIME

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We calculate the static Wilson loop from string/gauge correspondence to obtain the $Q\bar{Q}$ potential in nonrelativistic quantum field theory, i.e. CFT with Galilean symmetry. We analyze the convexity conditions[13] for $Q\bar{Q}$ potential in this theory, and obtain restrictions for the acceptable dynamical exponent z.

Keywords: Wilson loop; Galilean symmetry; $Q\bar{Q}$ pair; potential; holography.

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1. Introduction

It has been shown by Maldacena that large-N superconformal gauge theories have a dual description in terms of string theory in AdS space.[1] This proposal was realized by Maldacena to compute the energy between quark (Q) and anti-quark ($\bar{Q}$) pairs.[2] His method was to calculate expectation values of an operator similar to the Wilson loop in the large-N limit of field theories. Maldacena's idea was improved later by Rey, Theisen, and Yee.[3] It turns Wilson loop into a physical gauge-invariant property that can be read from the string picture. The $Q\bar{Q}$ energy in the large-N superconformal $\mathcal{N} = 4$ Yang–Mills theory can be obtained from the Wilson loop of the corresponding string in AdS space. It is proposed that quark and anti-quark pairs live on the boundary, connected by a U-shaped string in the bulk. In the discussion on this spacetime, the energy has a non-confining Coulomb-like behavior, as expected for a conformal field theory. Later this approach was applied to many other spaces and models, as summarized in Ref. 4.

Recently, gravity duals for a certain Galilean-invariant conformal field theory has attracted some attention in theoretical high energy physics community.[5–9] A special case when we take the dynamical exponent $z = 2$ of this theory (whose isometry is the Schrödinger group $Sch(d-1)$) is considered to be the basis in constructing duality between gravity and unitary Fermi gas. However, our interest in this paper is the theory with an arbitrary dynamical exponent z, i.e. Galilean invariant CFT. In this general scheme, one can discuss the nonrelativistic version of the AdS/CFT dictionary, i.e. the operator-state correspondence between the particle on the boundary and the string in the bulk. Scaling transformation in this nonrelativistic conformal symmetry can be written as[8–10]

$$x^i \to \lambda x^i \,, \qquad t \to \lambda^z t \,. \tag{1}$$

The asymptotic metric in this case can be written as

$$ds^2 = \frac{R^2}{r^2}\left(-\frac{dt^2}{r^{2(z-1)}} + dt\,d\xi + (dx^i)^2 + dr^2\right) + ds^2_{X_5}\,, \tag{2}$$

where R is the characteristic radius of spacetime, ξ is a compact light-like coordinate, x^i for $i = 1, \ldots, d$ together with t are the spacetime coordinates on the boundary where (2) is mapped at $r = 0$, and finally $ds^2_{X_5}$ is the metric of a suitable internal manifold geometry which allows (2) to be a solution of the supergravity equations of motion. The extra dimension ξ is usually associated with quantum numbers interpreted as the particle number. However, the relation between translation in ξ and its interpretation as particle number operator is still an unclear topic.[11,12] Thus we just set this time-like extra dimension ξ to be constant.

The holographic Wilson loop in nonrelativistic CFT had been studied by Klusoň in Ref. 11. He assumed general time dependence of ξ and also the moving $Q\bar{Q}$ pair cases in the context of nonrelativistic quantum field theory. His study was devoted to the spacetime with Galilean symmetry.[a] Nevertheless, he still does not include analysis of convexity conditions (12) and (13) yet. One needs to verify these conditions in $Q\bar{Q}$ potential discussions to make sure that the corresponding potential function $V(L)$ is a monotone non-decreasing and convex function of the separation L. The goal of this paper is to verify these conditions for $Q\bar{Q}$ potential, which is obtained by calculating the Wilson loop in the string picture in Galilean spacetime. Furthermore, we would like to see the restrictions which may appear for acceptable dynamical exponent z.

This paper is organized as follows. In Sec. 2, we will perform calculations to acquire the $Q\bar{Q}$ potential energy in Galilean spacetime. In Sec. 3, we will derive some conditions for acceptable z due to convexity inequality. Finally in Sec. 4, there is a summary of our findings.

2. $Q\bar{Q}$ Potential in Nonrelativistic CFT with Galilean Symmetry

We will start with the Nambu–Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma\, \sqrt{-\det G_{MN}\partial_\alpha x^M \partial_\beta x^N} \tag{3}$$

for metric (2) where $x^M = (t, r, \xi, x^i)$, G_{MN} is spacetime metric in (2), and impose suitable ansatzs in describing static strings, i.e. $t = x^0 = \tau$, $r = r(\sigma)$, $x = x(\sigma)$,

[a]From now on this will be abbreviated as Galilean spacetime.

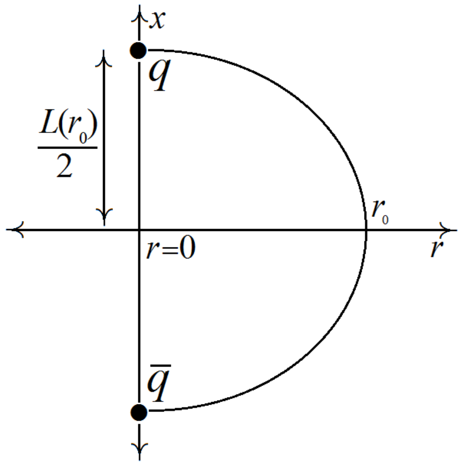

Fig. 1. $Q\bar{Q}$ pair on the boundary as each ends of string.

and ξ = constant. Klusoň in Ref. 11 has considered a more general case for an extra time-like dimension ξ as a τ-dependent variable, but we can simply set ξ to be constant (for example as discussed in Ref. 10) since the $Q\bar{Q}$ potential would depend on their separation distance[b] only. The corresponding action can be written as

$$S = -\frac{T}{2\pi\alpha'}\int d\sigma\,\sqrt{f^2(r)((r')^2+(x')^2)} \tag{4}$$

for $f(r) = R^2 r^{-(z+1)}$ and we have used $(\ )' \equiv \partial_\sigma(\ )$. Variable T in (4) is the loop period and can be written this way due to the time translation invariance of action (3) for metric (2). We have followed a standard prescription that has been used in some literature, for example Refs. 4, 14–18, in obtaining the action (4) as well as the corresponding $Q\bar{Q}$ potential as a function of $Q\bar{Q}$ pair's distance. Though the metric (2.1) is not diagonal, but action (4) leads us to a problem of Wilson loop computation which can be started by finding a geodesic in the effective two-dimensional geometry[18]

$$(ds_{\text{eff}})^2 = f^2(r)(dx^2 + dr^2)\,. \tag{5}$$

The equation of motion (geodesic line) from (4) is

$$\frac{dx}{dr} = \pm\frac{f(r_0)}{\sqrt{f^2(r)-f^2(r_0)}}\,. \tag{6}$$

r_0 is the maximum position of the U-shaped string with respect to the r-coordinate (bulk radius, see Fig. 1). From (6) one can obtain the separation distance of quark and anti-quark on the boundary, by integrating the geodesic with respect to r. Since

[b] A distance between Q and $\bar{Q}$ in our (3+1)-dimensional world, i.e. on the boundary of the Galilean bulk, see Fig. 1.

the boundary is at $r = 0$, then the separation as the function of r_0 can be obtained by the following integration

$$L(r_0) = 2\int_0^{r_0} \frac{f(r_0)}{\sqrt{f^2(r) - f^2(r_0)}} dr \,. \tag{7}$$

Related to the expression for the $Q\bar{Q}$ separation above, one may provide such an illustration as depicted in Fig. 1.

Inserting $f(r) = R^2 r^{-(z+1)}$ to (7) and using the beta function in our computation give the following exact result

$$L(r_0, z) = 2\int_0^{r_0} \frac{r^{z+1}}{\sqrt{r_0^{2z+2} - r^{2z+2}}} = \frac{2r_0\sqrt{\pi}\,\Gamma\left(\frac{z+2}{2z+2}\right)}{\Gamma\left(\frac{1}{2z+2}\right)} \,. \tag{8}$$

Then we follow a general prescription in Refs. 4, 15, 17 and 18 to compute the energy between quark and anti-quark. We have a general form of total $Q\bar{Q}$ energy as

$$E(r_0) = \frac{1}{\pi\alpha'}\int_0^{r_0} \frac{f^2(r)}{\sqrt{f^2(r) - f^2(r_0)}} dr - 2m_Q \,, \tag{9}$$

where m_Q is considered as the energy of non-interacting quark.[14,15,17,18] Thus the $Q\bar{Q}$ potential can be written as

$$\begin{aligned} V_{Q\bar{Q}}(r_0) &= E(r_0) - 2m_Q \\ &= \frac{1}{\pi\alpha'}\int_0^{r_0} \frac{f^2(r)}{\sqrt{f^2(r) - f^2(r_0)}} dr \end{aligned} \tag{10}$$

which can also be computed by the use of beta function. The potential is

$$V_{Q\bar{Q}}(r_0, z) = 2R^2 r_0^{z+1}\int_0^{r_0} \frac{dr}{r^{z+1}\sqrt{r_0^{2z+2} - r^{2z+2}}} = \frac{2R^2\sqrt{\pi}}{r_0^z(2z+2)} \frac{\Gamma\left(\frac{-z}{2z+2}\right)}{\Gamma\left(\frac{1}{2z+2}\right)} \,. \tag{11}$$

In the next section we will see the compatibility of the potential (11) with convexity conditions.

3. Convexity Conditions and String Embeddings

There are some conditions that should be satisfied by any potential which describes interaction between quark and anti-quark whose name "convexity" conditions[13,18]

$$\frac{dV}{dL} > 0 \tag{12}$$

and

$$\frac{d^2V}{dL^2} \leq 0 \,. \tag{13}$$

Condition (12) means quark and anti-quark are attractive everywhere, and (13) tells us that the potential is a monotone non-increasing function of their separation. These conditions can be verified as follows:

$$\frac{dV_{Q\bar{Q}}(r_0,z)}{dL(r_0,z)} = \frac{dV_{Q\bar{Q}}(r_0,z)}{dr_0}\frac{dr_0}{dL(r_0,z)} = \frac{-zR^2}{r_0^{z+1}(2z+2)}\frac{\Gamma\left(\frac{-z}{2z+2}\right)}{\Gamma\left(\frac{z+2}{2z+2}\right)} > 0 \tag{14}$$

and

$$\begin{aligned}\frac{d^2V_{Q\bar{Q}}(r_0,z)}{dL(r_0,z)^2} &= \frac{d\left(\frac{dV_{Q\bar{Q}}(r_0,z)}{dL(r_0,z)}\right)}{dr_0}\frac{dr_0}{dL(r_0,z)}\\ &= \frac{zR^2}{4\sqrt{\pi}r_0^{z+2}}\frac{\Gamma\left(\frac{1}{2z+2}\right)\Gamma\left(\frac{-z}{2z+2}\right)}{\left(\Gamma\left(\frac{z+2}{2z+2}\right)\right)^2} \leq 0\,.\end{aligned} \tag{15}$$

The last two equations are inequalities for physically accepted z based on convexity conditions for the $Q\bar{Q}$ pair.

In Ref. 19, the authors present simple embeddings of duals for nonrelativistic critical points, where the dynamical critical exponent can take many values $z \neq 2$.[c] They find that $z = 1$ and $z \geq 3/2$ as the possible dynamical critical exponents that allow string embeddings in gauge/gravity dual picture. From their paper,[19] we could learn that our $f(r)$ would depend on the coordinates of the internal manifold X_5.[d] Hartnoll and Yoshida write the non-compact part of the metric which can accommodate a large number of values of z by the following ansatz[e]

$$ds^2 = \frac{R^2}{r^2}\left(-\frac{dt^2}{h^2(X_5)r^{2(z-1)}} + dt\,d\xi + (dx^i)^2 + dr^2\right) \tag{16}$$

which modifies our previous $f(r)$ from $R^2r^{-(z+1)}$ to $R^2r^{-(z+1)}h(X_5)^{-1}$. Nevertheless, the function $h(X_5)$ would not appear in (8) and (11). Thus our findings on the restrictions for z can be applied to the work of Hartnoll and Yoshida in Ref. 19. One can verify that conditions (14) and (15) are fulfilled for $z = 1$, and also for $z \geq 3/2$. The negativity of $\Gamma\left(\frac{-z}{2z+2}\right)$ for $z \geq 1$ guarantees both (14) and (15) are satisfied.

4. Summary

We have calculated the potential between Q and $\bar{Q}$ in the nonrelativistic quantum field theory by using the Wilson loop analysis in the gauge/gravity correspondence in the Galilean bulk. Our findings are inequalities (14) and (15) for physically acceptable dynamical exponent z from convexity conditions. Yoshida and Hartnoll[19] have found families of z for string embeddings in Galilean spacetime, i.e. $z = 1$ and $z \geq 3/2$, which agree with inequalities (14) and (15) above.

[c] I thank Koushik Balasubramanian for informing me this work.
[d] I thank the reviewer for pointing this out to me.
[e] We follow the form of metric by Balasubramanian and McGreevy.[9] $f(X_5)$ in Ref. 19 is $h^2(X_5)$ in this paper.

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